Trinity College

Semester One Examination, 2018

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 1,2

Section Two: Calculator-assumed

SOLUTIONS

Student number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	54	35
Section Two: Calculator-assumed	13	13	100	101	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (99 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(7 marks)**

(a) A body travels with a velocity $-35\mathbf{i} - 12\mathbf{j}$ ms^{-1}. Determine its speed and the bearing on which it is moving, assuming the positive y-axis to be due north. (3 marks)

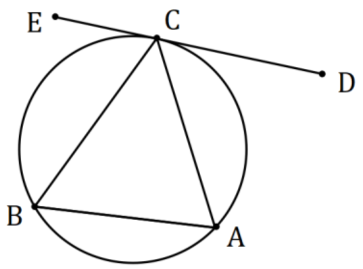
Solution
$\text{Speed} = \sqrt{(-35)^2 + (-12)^2} = 37\text{ m/s}$ $\text{Angle} = \tan^{-1}\left(\frac{-12}{-35}\right) = -161.1°$ $\text{Bearing} = 360n - (-161.1 - 90) = 251.1°$
Specific behaviours
✓ speed   ✓ angle   ✓ bearing

(b) Given that $\lambda(7\mathbf{i} - 10\mathbf{j}) + \mu(-9\mathbf{i} + 14\mathbf{j}) = -31\mathbf{i} + 46\mathbf{j}$, determine the values of λ and μ. (4 marks)

Solution
$7\lambda - 9\mu = -31$ $-10\lambda + 14\mu = 46$ $\lambda = -2.5$ $\mu = 1.5$
Specific behaviours
✓ equates $\mathbf{i}$-coefficients   ✓ equates $\mathbf{j}$-coefficients   ✓ value of λ ✓ value of μ

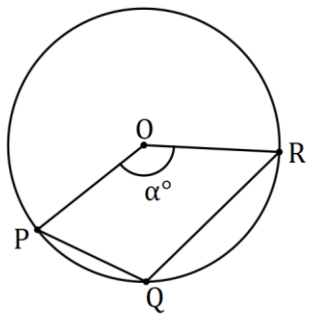
Question 10 **(6 marks)**

(a) In the diagram below, points A and B lie on a circle, DE is a tangent to the circle at C, $\angle DCA = 57°$ and $\angle CAB = 44°$. Determine the sizes of $\angle ABC, \angle BCE$ and $\angle BCA$. (3 marks)



Solution
$\angle ABC = \angle DCA = 57°$ $\angle BCE = \angle CAB = 44°$ $\angle BCA = \angle 180 - 57 - 44 = 79°$
Specific behaviours
✓ $\angle ABC$ ✓ $\angle BCE$ ✓ $\angle BCA$

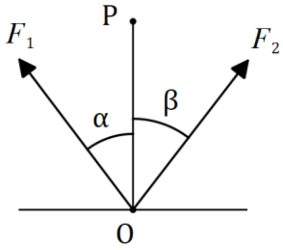
(b) In the next diagram, P, Q and R lie on a circle with centre O and $\angle POR = \alpha°$. Determine, with reasons, the size of $\angle PQR$ in terms of α. (3 marks)



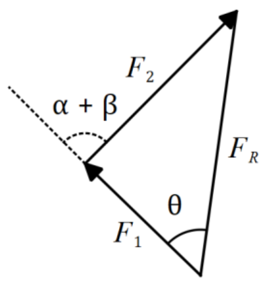
Solution
Reflex $\angle POR = x = 360 - \alpha$ (Angle sum at point) $\angle PQR = \frac{1}{2}x$ (Angle on arc at centre twice that on circumference) $\angle PQR = \frac{1}{2}(360 - \alpha) = 180 - \frac{1}{2}\alpha$
Specific behaviours
✓ expression for reflex angle ✓ reason ✓ $\angle PQR$ ✓ reason *(Or other methods)*

Question 11 **(8 marks)**

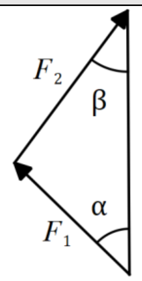
Two forces, $F_1 = 550 \text{ N}$ and $F_2 = 770 \text{ N}$, act on a body at O, and make angles of $\alpha = 33°$, and $\beta = 18°$ respectively with the vertical OP, as shown in the diagram below.



(a) Determine the magnitude of the resultant force and the angle it makes with the vertical. (5 marks)

Solution	
	$F_R = \sqrt{550 + 770^2 - 2(550)(770)\cos 129}$ $F_R = 1\,195\ N$ $\frac{\sin\theta}{770} = \frac{\sin 129}{1195.17}$ $\theta = 30°$ $\text{Angle} = 33° - 30° = 3°$
Specific behaviours	
✓ sketch with forces nose to tail ✓ indicates use of cosine rule for magnitude ✓ magnitude ✓ indicates use of sine rule for angle ✓ angle with vertical	

(b) The magnitude of F_1 is to be adjusted so that the direction of the resultant is vertical. Determine the required magnitude of F_1. (3 marks)

Solution	
	$\frac{\sin 18}{F_1} = \frac{\sin 33}{770}$ $F_1 = 437 \text{ N}$
Specific behaviours	
✓ sketch ✓ indicates use of sine rule ✓ magnitude	

Question 12 **(7 marks)**

The largest Australian family recently met with the largest American family. Between them, these two families had 39 children.

(a) Two of the children were chosen at random to feature in a TV documentary about the two families. Determine the number of different selections of two children that were possible. (1 mark)

Solution
$\binom{39}{2} = 741$
Specific behaviours
✓ correct number

(b) Prove that at least six of the children were born on the same day of the week. (3 marks)

Solution
Let the 7 days of the week be the pigeon-holes and the 39 children the pigeons. If 5 pigeons are placed in each of the 7 pigeon-holes, then there are still four left over, and so at least one of the pigeon-holes must have at least 6 pigeons (children).
Specific behaviours
✓ defines pigeons and pigeon-holes ✓ uses pigeon-hole principle ✓ states conclusion

There were more children in the American family than the Australian family and the American children all had blue, brown or hazel coloured eyes.

(c) Show that at least seven American children had the same eye colour. (3 marks)

Solution
$39 \div 2 = 19.5 \Rightarrow \text{minimum of } 20 \text{ from America}$ 3 eye colours are the pigeon-holes and 20 children are pigeons. If 6 pigeons placed in each pigeon hole there are 2 left over and so at least one of the pigeon-holes must have at least 7 pigeons (children with same coloured eyes).
Specific behaviours
✓ minimum number of American children ✓ defines pigeons and pigeon-holes ✓ uses pigeon-hole principle

Question 13 **(11 marks)**

(a) Simplify $(5\mathbf{a} - 3\mathbf{b}) \cdot (2\mathbf{a} + 4\mathbf{b})$ given that $|\mathbf{a}| = 6, |\mathbf{b}| = 5$ and vector $\mathbf{a}$ is parallel and in the same direction to vector $\mathbf{b}$. (4 marks)

Solution
$(5\mathbf{a} - 3\mathbf{b}) \cdot (2\mathbf{a} + 4\mathbf{b}) = 10\mathbf{a}\cdot\mathbf{a} + 20\mathbf{a}\cdot\mathbf{b} - 6\mathbf{b}\cdot\mathbf{a} - 12\mathbf{b}\cdot\mathbf{b}$ $= 10a^2 + 20ab - 6ab - 12b^2$ $= 10(36) + 14(30) - 12(25)$ $= 480$
Specific behaviours
✓ expands scalar product   ✓ indicates $\mathbf{a}\cdot\mathbf{b} = ab$ ✓ substitutes magnitudes   ✓ simplifies

(b) With respect to the origin O, points A and B have position vectors 2**i** +3**j** and -**i** + 5**j**. Determine $\overrightarrow{OC}$ if $\overrightarrow{AC} = 2\overrightarrow{AB}$. (4 marks)

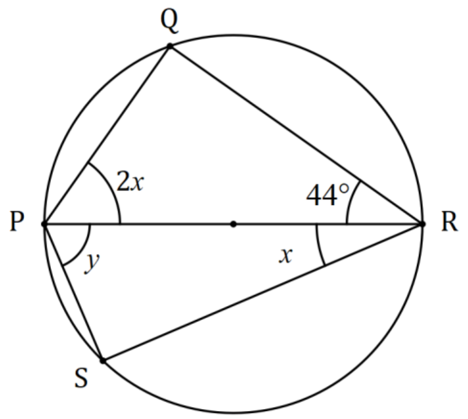
Solution
$\overrightarrow{AB} = \begin{pmatrix} \mathbf{-1} \\ \mathbf{5} \end{pmatrix} - \begin{pmatrix} \mathbf{2} \\ \mathbf{3} \end{pmatrix} = \begin{pmatrix} \mathbf{-3} \\ \mathbf{2} \end{pmatrix}$ $\overrightarrow{AC} = 2\overrightarrow{AB} = \begin{pmatrix} \mathbf{-6} \\ \mathbf{4} \end{pmatrix}$ $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$ $\begin{pmatrix} -6 \\ 4 \end{pmatrix} = \overrightarrow{OC} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\overrightarrow{OC} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$
Specific behaviours
✓ determines vectors AB   ✓ determines vector AC   ✓ determines vector equation involving vector OC   ✓ determines vector OC

(c) Points A, B and C have position vectors $(12, 7), (17, 3)$ and $(2, 15)$ respectively. Use a vector method to show that A, B and C are collinear. (3 marks)

Solution
$\overrightarrow{BA} = \begin{pmatrix} 12 \\ 7 \end{pmatrix} - \begin{pmatrix} 17 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$ $\overrightarrow{CA} = \begin{pmatrix} 12 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 15 \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \end{pmatrix}$ $\overrightarrow{BA} = -\frac{1}{2}\overrightarrow{CA}$ Hence points are collinear as $\overrightarrow{BA}$ and $\overrightarrow{CA}$ are parallel and both pass through the point A.
Specific behaviours
✓ determines $\overrightarrow{BA}$ and $\overrightarrow{CA}$ ✓ shows parallel   ✓ makes statement, mentioning common point.

Question 14 **(8 marks)**

(a) Determine the size of angles x and y in the diagram below, where Q and S lie on the circumference of the circle with diameter PR. (3 marks)



Solution
$2x + 44 = 90$ $x = 23°$ $y = 90 - 23 = 67°$
Specific behaviours
✓ uses angle in semi-circle   ✓ value of x ✓ value of y

(b) Triangle ABC has sides of length $AB = 4$ cm, $BC = 8$ cm and $AC = 7$ cm. Prove, using the method of contradiction, that if BC is a diameter of a circle then A does not lie on the circumference of the circle. (5 marks)

Solution
Assume that A does lie on the circumference of the circle.   Then $\angle BAC = 90°$ (angle in semi-circle)   Hence $AB^2 + AC^2 = BC^2$ (Pythagoras' theorem)   But $4^2 + 7^2 = 65 \neq 64 = 8^2$.   Hence assumption must be incorrect and so A does not lie on the circumference.
Specific behaviours
✓ assumes contradictory statement   ✓ states angle in semi-circle   ✓ deduces relationship between sides   ✓ shows contradiction   ✓ summarises

Question 15 **(9 marks)**

(a) Determine the number of integers between 1 and 500 that are divisible by 6 or 7. (4 marks)

Solution
$500 \div 6 = 83.3 \ldots \Rightarrow 83 \text{ divisible by } 6$ $500 \div 7 = 71.4 \ldots \Rightarrow 71 \text{ divisible by } 7$ $500 \div 42 = 11.9 \ldots \Rightarrow 11 \text{ divisible by both}$ $n = 83 + 71 - 11 = 143$
Specific behaviours
✓ divisible by 6 & 7   ✓ divisible by 42   ✓ use of inclusion-exclusion principle   ✓ correct number

(b) A pigeon fancier has 3 Fantail, 5 Carrier, 6 Archangel and 8 Dragoon pigeons and must choose four of them to enter in a local show. Determine the number of different ways the four pigeons can be chosen if

(i) there are no restrictions. (1 mark)

Solution
$\binom{22}{4} = 7\,315$
Specific behaviours
✓ correct number

(ii) the fancier decides to take one of each breed. (2 marks)

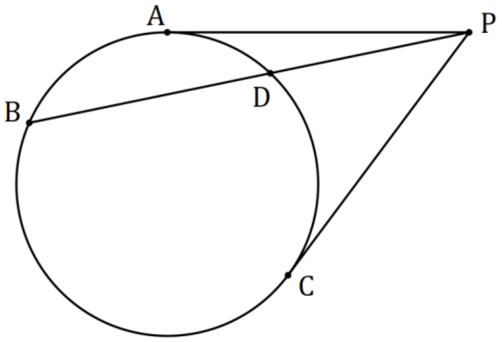
Solution
$\binom{3}{1} \times \binom{5}{1} \times \binom{6}{1} \times \binom{8}{1} = 720$
Specific behaviours
✓ uses multiplication principle   ✓ correct number

(iii) the fancier decides to take at least three Carrier pigeons. (2 marks)

Solution
$\binom{5}{3}\binom{17}{1} + \binom{5}{4}\binom{17}{0} = 170 + 5 = 175$
Specific behaviours
✓ indicates two cases   ✓ correct number

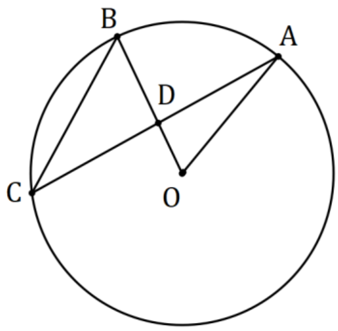
Question 16 **(9 marks)**

(a) In the diagram below, PA and PC are tangents to the circle, with $PA = 58$ cm. Secant PB cuts the circle at D, so that $PD = 40$ cm. Determine the lengths of PC and BD. (4 marks)



Solution
$PC = PA = 58$ cm $PD \times PB = AP^2$ $40(40 + BD) = 58^2$ $BD = 44.1$ cm
Specific behaviours
✓ value of PC ✓ indicates use of tangent-secant theorem ✓ equation for BD ✓ value of BD

(b) In the diagram below, A, B and C lie on the circumference of the circle with centre O, with AC intersecting OB at D. Prove that $\angle DAO = \angle DBC - \angle DCB$. (5 marks)



Solution
$\angle DAO + \angle DOA = \angle BDA = \angle DBC + \angle DCB$ (sum of exterior angles equal) But $\angle DOA = \angle BOA = 2\angle ACB = 2\angle DCB$ (angle at centre-circumference) Hence $\angle DAO + 2\angle DCB = \angle DBC + \angle DCB$ And so $\angle DAO = \angle DBC - \angle DCB$
Specific behaviours
✓ derives first equation ✓ reasoning for first equation ✓ uses angle at centre-circumference ✓ substitutes ✓ simplifies

Question 17 **(9 marks)**

Three vectors are $\mathbf{u} = a\mathbf{i} + b\mathbf{j}, \mathbf{v} = \mathbf{i} + 5\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} - 3\mathbf{j}$.

(a) Determine the vector projection of $\mathbf{w}$ on $\mathbf{v}$ in exact form. (3 marks)

Solution
$[\mathbf{w}\cdot\mathbf{v}] \times \frac{\mathbf{v}}{\lvert\mathbf{v}\rvert^2} = \left[\begin{pmatrix}2\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\5\end{pmatrix}\right] \times \frac{1}{26}\begin{pmatrix}1\\5\end{pmatrix}$ $= -\frac{1}{2}\mathbf{i} - \frac{5}{2}\mathbf{j}$
Specific behaviours
✓ indicates suitable form of projection   ✓substitutes correct values   ✓ solution in exact form

(b) Determine the scalar projection of $\mathbf{v}$ on $\mathbf{w}$**.** (2 marks)

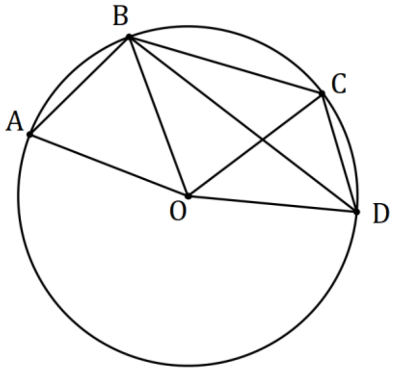
Solution
$\frac{[\mathbf{w}\cdot\mathbf{v}]}{\hat{\mathbf{w}}} = \frac{\left[\begin{pmatrix}2\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\5\end{pmatrix}\right]}{\sqrt{13}}$ =- $\sqrt{13}$
Specific behaviours
✓ indicates suitable form of projection   ✓ determine solution

(c) If $\mathbf{u}$ is perpendicular to $\mathbf{v}$ and has the same magnitude as $\mathbf{w}$, determine the exact values of the coefficients a and b. (4 marks)

Solution
$a^2 + b^2 = (2)^2 + (-3)^2 = 13$ $a + 5b = 0$ Using CAS, $a = \frac{5\sqrt{2}}{2}$ and $b = -\frac{\sqrt{2}}{2}$ or $a = -\frac{5\sqrt{2}}{2}$ and $b = \frac{\sqrt{2}}{2}$
Specific behaviours
✓ equation from magnitudes   ✓ equation from perpendicular   ✓ one solution   ✓ both solutions

Question 18 **(7 marks)**

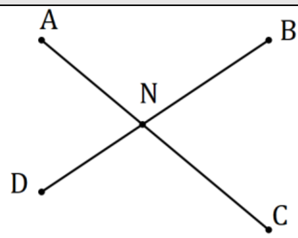
(a) In the diagram below, points B and C lie on the minor arc AD of the circle with centre O. The lengths of chords AB and CD are congruent, $\angle BOC = 59°$ and $\angle AOD = 173°$. Determine the size of $\angle CBD$. (3 marks)



Solution
$\angle AOD = \angle COD = \dfrac{173 - 59}{2} = 57°$ $\angle CBD = \dfrac{1}{2}\angle COD = 28.5°$
Specific behaviours
✓ indicates equal angles on equal chords   ✓ size of $\angle COD$ ✓ size of $\angle CBD$

(b) Line segment AC intersects line segment BD at N. Given that AC and BD are non-parallel and the lengths AN, AC, BN and BD are 35, 47, 59 and 67 cm respectively, explain whether the points A, B, C and D are concyclic. (4 marks)

Solution



$CN = 47 - 35 = 12, DN = 67 - 59 = 8$

$AN.CN = BN.DN$

$35 \times 12 = 420$ but $59 \times 8 = 472$

Not concyclic, as interval lengths do not satisfy the intersecting chord theorem.

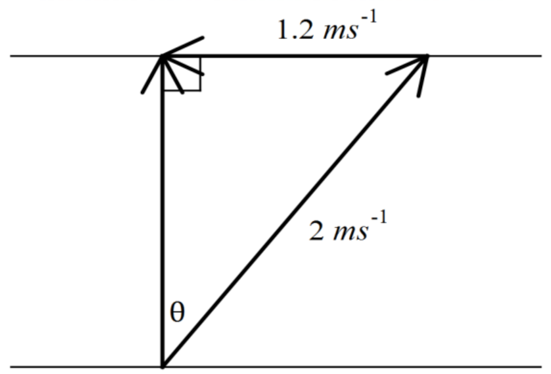
Specific behaviours

✓ sketch
✓ uses correct chord lengths
✓ uses property of intersecting chords
✓ explanation

Question 19 **(6 marks)**

Daivik can row a boat at 2 m/s in still water. He sets out to cross a river which is running at 1.2 m/s.

(a) In which direction should he row so that he crosses at right angles to the bank? (3 marks)

Solution	
	$\sin\vartheta = \frac{1.2}{2}$ $\vartheta = 36.9°$ $36.9°$ *from directly straight across*
Specific behaviours	
✓ sketch ✓ uses sin ✓ determines direction	

(b) Daivik decides to row so that he gets to the opposite bank as quickly as possible.

(i) What direction must he row in order to get to the opposite bank in the least time? (1 mark)

Solution
Directly across the river
Specific behaviours
✓ uses multiplication principle

(ii) If the river is 50 m wide, how long will it take to get to the other side? (1 mark)

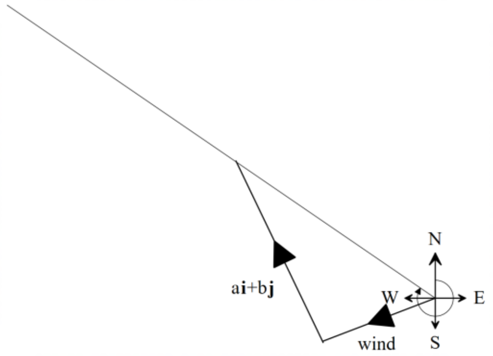
Solution
$\frac{50}{2} = 25 \text{ sec}$
Specific behaviours
✓ determines time

(iii) Where will Daivik land on the opposite bank? (1 mark)

Solution
$25 \times 1.2 = 30\ m\ along\ opposite\ bank$
Specific behaviours
✓ determine distance along bank

Question 20 **(7 marks)**

A small boat leaves jetty A to travel to jetty B, 377 m away on a bearing of $310°$. A steady current of 1.5 ms^{-1} runs in the river between the jetties on a bearing $250°$. If the small boat travels at a constant speed of 4.6 ms^{-1}, determine the bearing it should steer to reach jetty B and how long the journey will take.

Solution

$\begin{pmatrix} 1.5 \\ -160° \end{pmatrix} = \begin{pmatrix} -1.41 \\ -0.51 \end{pmatrix}$ $\begin{pmatrix} 377 \\ 140° \end{pmatrix} = \begin{pmatrix} -288.8 \\ 202.33 \end{pmatrix}$
$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} -1.41 \\ -0.51 \end{pmatrix} = \lambda \begin{pmatrix} -288.8 \\ 202.33 \end{pmatrix}$
$a = 1.41 - 288.8\lambda$
$b = 0.51 + 202.33\lambda$
$a^2 + b^2 = 4.6$
$\lambda = 0.0137, \quad a = -2.55, \quad b = 3.83$
$Bearing\ 326.4°$
$Time = \dfrac{1}{\lambda} = 73\text{sec}$
Specific behaviours
✓ diagram ✓ converts polar coords to rectangular coords ✓ writes equations ✓ solves for λ ✓ solves for a, b ✓ correct time ✓ correct bearing

Question 21 **(7 marks)**

A child is playing with thirteen coloured cubes, all the same size. There are six pink cubes, three navy and one each of red, blue, orange and green.

(a) If the child stacks cubes one on top of another to make a column, determine the number of different coloured columns that can be made using

(i) all the red, blue and green cubes. (1 mark)

Solution
$3! = 6$
Specific behaviours
✓ number

(ii) all the pink, red and orange cubes. (2 marks)

Solution
$\dfrac{(6+1+1)!}{6!} = \dfrac{8!}{6!} = 56$
Specific behaviours
✓ numerator ✓ correct number

(iii) all the cubes. (2 marks)

Solution
$\dfrac{13!}{6!\,3!} = 1\,441\,440$
Specific behaviours
✓ expression ✓ correct number

(b) If all but one of the cubes are used to make a column, determine the number of different coloured columns that can now be made. Justify your answer. (2 marks)

Solution
$1\,441\,400$ columns All the columns 13 tall with a pink on top must have a difference in the 12 cubes beneath and so if the top pink is removed, the remaining columns will still be different. The same is true for columns with other coloured top cubes, and the remaining 12 tall columns will have one less cube of the top colour and so must be different to all other columns. So, no change.
Specific behaviours
✓ correct number ✓ justification

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Supplementary page

Question number: _________